Two-Variable Logic with Two Order Relations* (Extended Abstract)

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Abstract. The finite satisfiability problem for two-variable logic over structures with unary relations and two order relations is investigated. Firstly, decidability is shown for structures with one total preorder relation and one linear order relation. More specifically, we show that this problem is complete for EXPSPACE. As a consequence, the same upper bound applies to the case of two linear orders. Secondly, we prove undecidability for structures with two total preorder relations as well as for structures with one total preorder and two linear order relations. Further, we point out connections to other logics. Decidability is shown for two-variable logic on data words with orders on both positions and data values, but without a successor relation. We also study "partial models" of compass and interval temporal logic and prove decidability for some of their fragments.

1 Introduction

First-order logic restricted to two-variables (*two-variable logic* or FO^2 in the following) is generally known to be reasonably expressive for many purposes and to behave moderately with respect to the possibility of testing satisfiability. As opposed to full first-order logic, its satisfiability and its finite satisfiability problem are decidable [Mor75], in fact they are NEXPTIME-complete [GKV97]. However, there are some simple properties like transitivity of a binary relation that cannot be expressed in FO^2. As a consequence, if one is interested in models with particular properties it might be the case that these properties cannot be described in an FO^2 formula and thus one is interested in the question whether a formula has a model with these additional properties. Two particular examples are the inability to express that a binary relation is a linear order or that it is an equivalence relation, both are actually due to the inability to axiomatize transitivity.

In [Ott01] it is shown that it is decidable in NEXPTIME whether a given FO^2 sentence has a model (or whether it has a finite model) in which a particular relation symbol is interpreted by a linear order. On the other hand, in the

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presence of eight binary symbols that have to be interpreted as linear orders it is undecidable. Note that in these results the formulas might use further relation symbols for which the possible interpretations are unrestricted. In [KO05] it is shown[1] that finite satisfiability is NEXPTIME-complete over structures with one equivalence relation and undecidable over structures with three equivalence relations. In [KT09] it is shown that over structures with two equivalence classes the problem is decidable in triply exponential nondeterministic time.

In this paper we study two-variable logic over structures with linear orders and total preorders. A total preorder $\precsim$ is basically an equivalence relation $\sim$ whose equivalence classes are ordered by $\prec$. Total preorders can therefore encode equivalence relations as well as linear orders and in this sense they generalize both these kinds of relations. It should be stressed that in our results, formulas can refer to an arbitrary number of additional unary relations but they are *not* allowed to refer to arbitrary non-unary relations besides the orders that are explicitly mentioned.

Our motivation stems from the context of so-called data words. A *data word* is a word, that is, a finite sequence of symbols from a finite alphabet, but besides a symbol, every position also carries a value from a possibly infinite domain. The interest in data words and data trees comes on one hand from applications in database theory, where XML documents can be modeled by data trees in which the symbols correspond to the tags and the data values to text or number values. On the other hand, (infinite) data words can also be considered as traces of a computation in a distributed environment, where symbols correspond to states of processes and the data values encode process numbers. Recently many logics and automata models have been considered for data words and data trees (see [Seg06] for a gentle introduction).

First-order logic on data words is undecidable in general (with a linear order on positions and equality on data values), even for formulas with three variables [BMS$^+$06]. Whether two-variable logics is decidable depends on the way the order of positions in the word is represented and on the ability to compare data values. In [BMS$^+$06] it is shown that finite satisfiability of FO^2 over data words is NEXPTIME-complete if only a linear order on the positions is given and data values can only be compared with respect to equality. The attentive reader might have noticed that this is just the setting of structures with one linear order and one equivalence relation (and some unary relations). If the successor relation (+1) on the positions is also available the problem remains decidable but the complexity is unknown (but basically equivalent to the open complexity of Petri net reachability). The same paper shows that if furthermore data values can be compared with respect to a linear order (by a binary relation stating that "the value at position x is smaller than the value at y") the logic immediately becomes undecidable. The latter case draws a link between data words and structures with linear orders and total preorders. As already mentioned, the representation of words usually involves a linear order on the positions. A linear order on data

[1] As this paper only deals with finite structures we henceforth only mention results on finite satisfiability.

values induces a total preorder on the positions. Two positions can carry the same data value and therefore are considered "equivalent" with respect to their data value or one can carry a smaller data value than the other. It is exactly this setting which triggered our study of structures with a linear order, a total preorder and a number of unary relations.

Results. We show that finite satisfiability of two-variable logic over structures with a linear order, a total preorder and unary relations is EXPSPACE-complete. Thus, finite satisfiability of FO^2 over data words with a linear order on the positions (but no successor relation) and a linear order on the data values can also be decided in exponential space. As it can be expressed in FO^2 that a total preorder is a linear order, the corresponding problem with two linear orders (and no total preorder) is solvable in EXPSPACE, thus the gap left in the work of Otto [Ott01] is narrowed. In contrast, finite satisfiability of FO^2 over structures[2] with two total preorders and over structures with two linear orders and a total preorder is undecidable.

The upper bound in the case of a linear order and a total preorder is by a reduction to finite satisfiability of *semi-positive* FO^2 sentences over sets of labeled points in the plane, where points can be compared by their relative position with respect to the directions $\nwarrow, \nearrow, \swarrow, \searrow, \leftarrow, \rightarrow$. In semi-positive formulas, negated binary atoms are not allowed in the "immediate scope" of an existential quantifier. This satisfiability problem in turn is reduced to a constraint problem for labeled points in the plane. The lower bound is by a reduction from exponential width corridor tiling.

Furthermore, we use the result on semi-positive two-variable logics over labeled point sets to obtain complexity bounds for the problem to decide the existence of partial models for some fragments of compass logic and interval logic.

Organization. After some basic definitions in Section 2 we show the main results on ordered structures in Section 3 and discuss the applications for data words, compass logics and interval logics in Section 4. We conclude with Section 5. Due to space restrictions, some proof details are deferred to the full version of the paper.

Related work. As mentioned before, also other logics for data words besides FO^2 have been studied. As an example we mention the "freeze"-extension of LTL studied, e.g. in [DL09] and [FS09]. The latter paper is more closely related to our work as it considers a restriction of LTL without the X-operator. We are aware that Amaldev Manuel has recently proved decidability and undecidability results for FO^2-logic over structures with orders [Man10]. However, as in his results structures always contain at least one successor relation neither results nor techniques translate from his work to ours nor vice versa. Besides the relations to compass logic and interval logic discussed above we conjecture that there are

[2] Additional unary relations are again allowed.

connections to spatial reasoning such as it is done in the context of Geographical Information Systems (for a survey, see [CH01]). Related work on compass and interval logic is mentioned in Section 4.

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2 Preliminaries

We consider two kinds of two-variable logics: over ordered structures and over labeled point sets in the plane. We first fix our notation concerning order relations. A *total preorder* $\precsim$ is a transitive, total relation, i.e. $u \precsim v$ and $v \precsim w$ implies $u \precsim w$ and for every two elements u, v of a structure $u \precsim v$ or $v \precsim u$ holds. A *linear order* $\leq$ is a total preorder which is antisymmetric, i.e. if $u \leq v$ and $v \leq u$ then $u = v$. Thus, the essential difference between a total preorder and a linear order is that the former allows that two distinct elements u, v are equivalent with respect to $\precsim$, that is, both $u \precsim v$ and $v \precsim u$ hold. Thus, a total preorder can also be viewed as an equivalence relation $\sim$ whose equivalence classes are strictly and linearly ordered by $\prec$. Clearly, every linear order is a total preorder.

We use binary relation symbols $\precsim, \precsim_1, \precsim_2, \ldots$ that are always interpreted by total preorders as well as binary relation symbols $\leq, \leq_1, \leq_2, \ldots$ that are always interpreted as linear orders.

In this paper, an *ordered structure* is a finite structure with non-empty universe and some linear orders, some total preorders and some unary relations. We always allow an unlimited number of unary relations and specify the numbers of allowed linear orders and total preorders explicitly. For example, by $FO^2(\leq_1, \precsim_1, \precsim_2)$ we denote the set of two-variable sentences over structures with one linear order, two total preorders and arbitrarily many unary relations. Furthermore, all two-variable formulas in this paper can refer to the equality relation $=$.

As mentioned before, we also consider formulas that express properties of sets of labeled points. Let $\mathcal{P} = \{e_1, \ldots, e_k\}$ be a set of propositions. A *$\mathcal{P}$-labeled point* p is a point in $\mathbb{Q}^2$ in which propositions $e_1, \ldots, e_k$ may or may not hold. We refer to the x-coordinate, the y-coordinate of a point p by $p.x$ and $p.y$, respectively, and write $p.e_i = 1$, if proposition e_i holds in p. We simply say *point* if $\mathcal{P}$ is understood from the context.

We write $p \nearrow q$ if $p.x < q.x$ and $p.y < q.y$, we write $p \nwarrow q$ if $p.x > q.x$ and $p.y < q.y$. Analogously for $p \searrow q$ and $p \swarrow q$. We write $p \rightarrow q$ if $p.y = q.y$ and $p.x < q.x$ and likewise $p \leftarrow q$. Analogously for $\uparrow$ and $\downarrow$. Let $\mathcal{D} = \{\nwarrow, \nearrow, \swarrow, \searrow, \leftarrow, \rightarrow, \uparrow, \downarrow\}$ denote the set of *directions*. We denote the set $\{\nwarrow, \nearrow, \swarrow, \searrow, \leftarrow, \rightarrow\})$ by $\mathcal{D}_-$

A set $\mathcal{O} \subseteq \mathcal{D}$ is *symmetric* if it contains, for each direction also the opposite direction, e.g., if it contains $\nearrow$ then also $\swarrow$. We will only consider symmetric sets of directions. For a symmetric set $\mathcal{O} \subseteq \mathcal{D}$, $\mathrm{FO}^2(\mathcal{O})$ denotes two-variable logic sentences with binary atoms using directions from $\mathcal{O}$. Such a sentence is *semi-positive* if it is in negation normal form (NNF) and fulfils the following condition: whenever a negated binary atom $\neg(x \circ_d y)$ with $\circ_d \in \mathcal{O}$ occurs in the scope of an $\exists$-quantifier, there is a $\forall$-quantifier in the scope of this quantifier and the atom in turn is in the scope of this $\forall$-quantifier. In this sense, no negated binary atom is in the "immediate scope" of an $\exists$-quantifier. Negated atoms of the form $x \neq y$ are allowed in the immediate scope of $\exists$-quantifiers.

We interpret $\mathrm{FO}^2(\mathcal{O})$ sentences over non-empty sets of $\mathcal{P}$-points in $\mathbb{Q}^2$, where $\mathcal{P}$ has a proposition e_i, for every unary relation symbol U_i.

3 Two-variable logic with a total preorder and a linear order

In this section we show the main result of the paper.

Theorem 1. *Finite satisfiability of* $\mathrm{FO}^2(\leq, \precsim)$ *is* EXPSPACE-*complete.*

The upper bound of Theorem 1 is shown in three steps. First, we reduce finite satisfiability of $\mathrm{FO}^2(\leq, \precsim)$-sentences in polynomial time to finite satisfiability of semi-positive $\mathrm{FO}^2(\mathcal{D}_-)$-sentences. Next, we reduce finite satisfiability for semi-positive $\mathrm{FO}^2(\mathcal{O})$, for every symmetric set $\mathcal{O}$ of directions, in exponential time to the two-dimensional labeled point problem ($\mathrm{2LPP}(\mathcal{O})$) which is defined below. Finally, we show that $\mathrm{2LPP}(\mathcal{D}_-)$ can be solved in polynomial space. The lower bound is by a reduction from exponential width corridor tiling. As a FO^2-formula can express that a total preorder is actually a linear order, we get the following.

Corollary 2. *Finite satisfiability of* $\mathrm{FO}^2(\leq_1, \leq_2)$ *is in* EXPSPACE.

3.1 From ordered structures to point sets

Proposition 3. *For each* $\mathrm{FO}^2(\leq, \precsim)$*-sentence* φ *a semi-positive* $\mathrm{FO}^2(\mathcal{D}_-)$*-sentence* φ' *can be computed in polynomial time that is equivalent with respect to finite satisfiability.*

Proof. We first explain, how finite point sets and structures with a linear order $\leq$ and a total preorder $\precsim$ can be translated into each other.

With every finite ordered structure S with domain D and relations $\leq$ and $\preceq$ and some unary relations one can associate a finite point set P in the plane and a bijection π such that

- $u < v$ in S if and only if $\pi(u) \nearrow \pi(v)$ or $\pi(u) \rightarrow \pi(v)$ or $\pi(u) \searrow \pi(v)$,
- $u \prec v$ in S if and only if $\pi(u) \nwarrow \pi(v)$ or $\pi(u) \nearrow \pi(v)$, and
- $u \sim v$ in S if and only if $\pi(u) \leftarrow \pi(v)$ or $\pi(u) \rightarrow \pi(v)$.

To this end, one can first assign to each element of D an x-value in increasing fashion with respect to $<$. Second, each element gets a y-value in accordance with $\precsim$. It is easy to see that in P there are no two points with the same x-value. Thus, our reduction has to guarantee that φ' only has models with this additional property. The translation in the other direction is analogous.

For the actual reduction, first of all, φ' has a conjunct χ that ensures that no two points are on the same vertical line. This can be easily expressed by $\forall x \forall y \bigvee_{\circ_d \in \mathcal{D}_-} x \circ_d y$. As we can assume that φ is given in NNF it suffices to explain how the possible negated or positive binary atoms of φ are translated. The translation is as follows.

φ	φ'
$x \leq y$	$x = y \vee x \nearrow y \vee x \rightarrow y \vee x \searrow y$
$\neg(x \leq y)$	$x \nwarrow y \vee x \leftarrow y \vee x \swarrow y$
$x \precsim y$	$x = y \vee x \rightarrow y \vee x \leftarrow y \vee x \nearrow y \vee x \nwarrow y$
$\neg(x \precsim y)$	$x \searrow y \vee x \swarrow y$

The correctness of this translation relies on the fact that χ ensures that no two points are on the same vertical line. □

Theorem 1 follows from Proposition 3 and the following result, the main technical contribution of this paper, which will be shown in the next two subsections.

Theorem 4. *Whether a semi-positive* $\mathrm{FO}^2(\mathcal{D}_-)$*-formula has a finite model, can be decided in exponential space.*

3.2 From two-variable logic to two-dimensional constraints

We next define 2LPP($\mathcal{O}$). For an alphabet Σ, a *Σ-labeled point* p is an element from $\mathbb{Q}^2 \times \Sigma$. The only difference between Σ-labeled points and $\mathcal{P}$-labeled points is that the former are labeled by one symbol from a finite alphabet whereas the latter carry several propositions.

An *existential constraint ($\exists$-constraint)* is a pair (σ, E) where $\sigma \in \Sigma$ and E is a possibly empty set of pairs $(\circ_d, \tau)$ with $\circ_d \in \mathcal{O} \cup \{*, \neq\}$ and $\tau \in \Sigma$. A *universal constraint ($\forall$-constraint)* is a tuple (σ, τ, O_1, O_2) where $\sigma, \tau \in \Sigma$, $O_1 \subseteq \mathcal{O} \cup \{=\}$ and $O_2 \subseteq \mathcal{O}$. A set M of Σ-labeled points *satisfies* an $\exists$-constraint (σ, E) if, for every $p \in M$ with $p.l = \sigma$ there is $q \in M$ such that, for some $(\circ_d, \tau)$ in E, $q.l = \tau$ and $p \circ_d q$. Here, $p * q$ is true for all p, q. It *satisfies* a $\forall$-constraint (σ, τ, O_1, O_2) if for all points $p, q \in M$ with $p.l = \sigma$, $q.l = \tau$ it holds $p \circ_d q$ for some $\circ_d \in O_1$ or $\circ_d \notin O_2$.

An input $L = (\Sigma, C_\exists, C_\forall)$ to the *two-dimensional labeled point problem (2LPP($\mathcal{O}$))* consists of an alphabet Σ, a set $C_\exists$ of existential constraints and a set $C_\forall$ of universal constraints and all constraints only use directions from $\mathcal{O}$. A non-empty finite set $M \subseteq \mathbb{Q}^2 \times \Sigma$ is a *solution* of L if M satisfies all constraints from $C_\exists$ and $C_\forall$.

Proposition 5. *Let $\mathcal{O} \subseteq \mathcal{D}$ be a symmetric set of directions. From every semi-positive* $\mathrm{FO}^2(\mathcal{O})$*-sentence φ an instance L of 2LPP($\mathcal{O}$) can be computed in exponential time such that φ has a finite model if and only if L has a solution.*

Proof. Let $\mathcal{O} \subseteq \mathcal{D}$ and φ be an $\mathrm{FO}^2(\mathcal{O})$-sentence.

First, φ can be translated into a formula φ' in Scott normal form (SNF)

$$\forall x \forall y\ \psi(x,y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y\ \psi_i(x,y),$$

that has a finite model if and only if φ has a finite model. The translation can be done in a way that ensures that

(1) ψ and all ψ_i are quantifier-free formulas,
(2) φ' is of linear size in the size of φ,
(3) negated binary atoms only occur in ψ.

In general, φ' uses more unary relation symbols than φ.

The translation is basically done as described in [GO99, Section 2.1]. However, some care is needed to guarantee condition (3). In the translation into SNF given in [GO99], for every quantified sub-formula $\psi(x)$ of the form $\exists y\ \psi_0$ or $\forall y\ \psi_0$ a unary relation P_ψ is introduced with the intention that every element a in a structure should obey $P_\psi(a) \Leftrightarrow \psi(a)$. The addition of such formulas would introduce a negation of ψ and therefore destroy condition (3). Fortunately, as we only consider formulas in NNF, it is sufficient that every element respects $P_\psi(a) \Rightarrow \psi(a)$ and therefore the introduction of negation can be avoided. To see that this suffices let us assume without loss of generality that φ is of the form $\exists x \chi$. If A is a model of φ then the relations P_ψ can be chosen such that $P_\psi(a) \Leftrightarrow \psi(a)$ holds, for every a and every subformula ψ of χ. On the other hand, by induction it can be shown that $P_\psi(a)$ only holds if $\psi(a)$ holds, thus the outermost formula $\exists x P_\chi(x)$ becomes true only if φ is true. Semi-positivity of φ ensures that no atoms $\neg(x \circ_d y)$ occur in any ψ_i.

In order to continue the translation into an 2LPP-instance, let ψ' be a disjunctive normal form of ψ. We can ensure that each disjunct of ψ' is the conjunction of a full atomic type σ for x, a full atomic type τ for y, and some (negated or positive) binary atoms. As the binary relations interpreting the directions in $\mathcal{O}$ are pairwise disjoint, we can assume that there is either (1) one positive binary atom or (2) a set of negated binary atoms of the form $x \circ_d y$ with $\circ_d \in \mathcal{O}$. For two positive atoms with different directions would evaluate to false and likewise negative atoms (with other directions) are redundant in the presence of a positive atom. Thus, by combining all clauses with the same σ and τ we get a set O_1 of "allowed directions" and a set O_2' of sets of forbidden directions. The set O_2' can be combined into one set O_2 of forbidden directions. It should be noted that O_2 could be the empty set. Altogether, we obtain a set $C_\forall$ of universal constraints equivalent to ψ. Clearly, the size of $C_\forall$ is at most exponential in φ.

By transforming the ψ_i into DNF and some additional simple steps, the second conjunct, $\bigwedge_{j=1}^{m} \forall x \exists y\ \psi_i(x,y)$, of φ' can be rewritten as

$$\forall x \bigwedge_{i=1}^{K} \big(\sigma_i(x) \Rightarrow \bigwedge_{j=1}^{m} \exists y \bigvee_{\ell=1}^{M} \psi_{ij\ell}\big),$$

where the σ_i describe pairwise distinct full atomic types and every $\psi_{ij\ell}$ is the conjunction of a full atomic type for y and (1) an atom $x \circ_d y$ with $\circ_d \in \mathcal{O} \cup \{\neq\}$ or (2) no further literals. The numbers K and M are at most exponential in $|\varphi|$. The semantics of this formula can be easily expressed by a set $C_\exists$ of $\exists$-constraints. For each $i \leq K$ and every $j \leq m$, the formulas $\psi_{ij\ell}$ can be combined into one $\exists$-constraint. Thus, $L = (\Sigma, C_\exists, C_\forall)$ is an instance of 2LPP($\mathcal{O}$) and a Σ-labeled point set is a solution to L if and only if the corresponding labeled point set $\mathcal{P}$ is a model of φ. Therefore L has a solution if and only if φ has a finite model. Furthermore, the size of L is at most exponential in $|\varphi|$ and the construction of L from φ can be done in exponential time. □

3.3 2LPP($\mathcal{D}_-$) is in PSPACE

Proposition 6. *Whether an instance L of 2LPP($\mathcal{D}_-$) has a solution can be decided in polynomial space.*

We basically show that every satisfiable formula has a model of exponential size and polynomial width.

To this end, we assume that every solution M of a labeled point problem comes with a partial *witness function* $w : M \times C_\exists \to M$ such that, for every point $p \in M$ and every $\exists$-constraint $c = (\sigma, E)$ with $p.l = \sigma$

- $w(p, c)$ exists, and
- there is some $(\circ_d, \tau)$ in E such that $p \circ_d w(p, c)$, and $w(p, c).l = \tau$.

We call the point $w(p, c)$ *witness* of p wrt c. A *horizontal line* $H(r)$ is a non-empty set of points $p \in M$ with $p.y = r$.

Lemma 7. *If M is a model for an instance of 2LPP($\mathcal{D}_-$) with alphabet Σ then every horizontal line of M contains at most $2|\Sigma|$ points.*

The proof uses the simple observation that if a horizontal line contains three points with the same label, only the two outermost points are needed as witnesses for other points.

Lemma 8. *Let L be an instance of 2LPP($\mathcal{D}_-$) and M a minimal solution to L. Then the number of horizontal lines that contain points from M is bounded by $(6|\Sigma| + 1)!2^{6|\Sigma|}$.*

Proof. For $r \in \mathbb{Q}$, we define the set $A(r)$ of ***a****vailable witness points* for r containing $H(r)$ and, for each $\sigma \in \Sigma$, σ-points with minimum and maximum x-value above $H(r)$ and σ-points with minimum and maximum x-value below $H(r)$. Note that $A(r)$ contains at most six points with a given σ and thus $|A(r)| \leq 6|\Sigma|$.

The *profile* Pro(r) is obtained from $A(r)$ as follows. First, for each point $p \in A(r)$ we construct a pair $(p.l, p.f)$, where $p.f =\uparrow$ if $p.y > r$, $p.f = \cdot$ if $p.y = r$ and $p.f =\downarrow$ if $p.y < r$. We partition this set into maximal sets of pairs

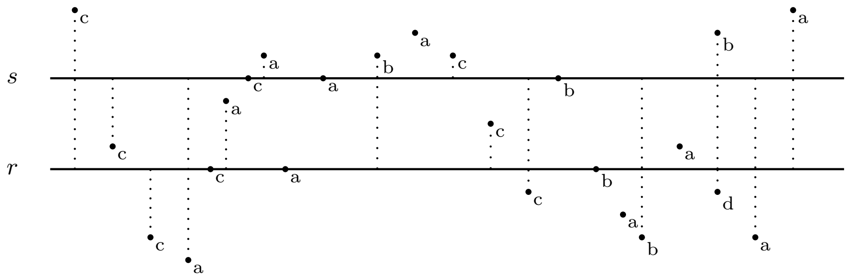

Fig. 1. Two horizontal lines with the same profile $(c,\uparrow)(c,\downarrow)(a,\downarrow)(c,\cdot)(a,\uparrow)(a,\cdot)(b,\uparrow)(c,\uparrow)(c,\downarrow)(b,\cdot)(b,\downarrow)\{(b,\uparrow),(d,\downarrow)\}(a,\downarrow)(a,\uparrow)$. Dotted lines indicate points contributing to the profiles.

resulting from points with the same x-value and order the partition by these x-values. We refer to Figure 1 for two example profiles. Singleton sets in profiles are written without braces. Thus, a profile is an ordered partition of a multiset of size $\leq 6|\Sigma|$. It is easy to see that there are less than $N =_{\text{def}} (6|\Sigma|+1)!2^{6|\Sigma|}$ different profiles. Thus, if M has more than N horizontal lines two of them must have the same profile.

Let us thus assume, towards a contradiction, that there exist $r < s$ such that there are points p, q in M with $p.y = r$ and $q.y = s$ and $\text{Pro}(r) = \text{Pro}(s)$. We let $S_{<r} =_{\text{def}} \{q \in \mathbb{Q}^2 \mid q.y < r\}$, $S_{=r} =_{\text{def}} \{q \in \mathbb{Q}^2 \mid q.y = r\}$, and $S_{>r} =_{\text{def}} \{q \in \mathbb{Q}^2 \mid q.y > r\}$.

We construct M' as follows (Figure 2 illustrates the construction).

- All points p from $M \cap S_{\geq s}$ are in M'.
- No point p from M with $r \leq p.y < s$ is in M'.
- Informally, the vertical stripe between p_i and p_{i+1} and below r is scaled proportionally and shifted to the vertical stripe between q_i and q_{i+1}. More precisely, let k be the number of points in $A(r)$ (and $A(s)$) and let $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ be the points from $A(r)$ and $A(s)$, respectively, sorted with respect to their x-coordinate (and thus in the same way as in $\text{Pro}(r)$ and $\text{Pro}(s)$, respectively). Let $p \in M \cap S_{<r}$ and let $i \in \{1, \ldots, k\}$ be such that $p_i.x \leq p.x < p_{i+1}.x$. It should be noted that there is no point p with $p.x < p_1.x$ or $p.x > p_k.x$. Then M' contains the point p' defined by
  - $p'.l = p.l$,
  - $p'.y = p.y$, and
  - $p'.x = q_i.x + \frac{p.x - p_i.x}{p_{i+1}.x - p_i.x}(q_{i+1}.x - q_i.x)$.

  We call p the source of p'.

  It could happen that a point below r has the same x-value as some point above r. To avoid that a point $p \notin A(r)$ below r has the same x-value as some point $s \notin A(s)$ above s, we finally shift every point $p \notin A(r)$ below r by ϵ in eastern direction, where ϵ is smaller than the minimal non-zero vertical distance of any two points. This final step ensures that in M' there are no two points with the same x-value if in M there were no such two points.

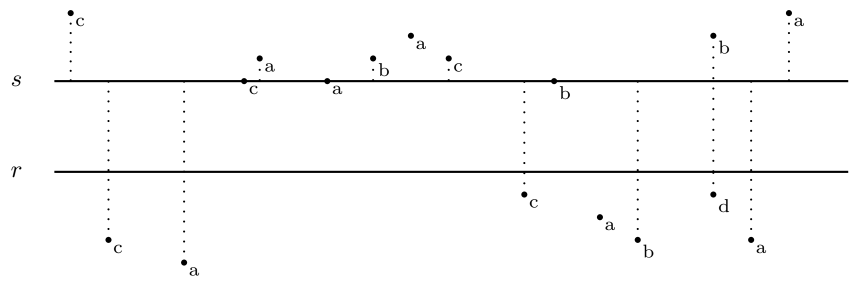

Fig. 2. Contraction of Figure 1. It should be noted that the lower a-point not in $A(r)$ is slightly shifted to the left and is now in the exact middle between the b-point on the upper horizontal line and the lower b-point.

Clearly, $|M'| < |M|$ as we removed (at least) the points of the horizontal line with y-coordinate r. Thus, to obtain a contradiction it is sufficient to show that M' is a solution to L.

To show that all $\forall$-constraints are satisfied it is sufficient to observe that if there are points p', q' in M' with $p' \circ_d q'$ for any $\circ_d \in \mathcal{D}$ then there are points p and q in M such that $p.l = p'.l$, $q.l = q'.l$ and $p \circ_d q$. This is clear for all pairs $p'q'$ of points above or on s as M and M' are identical here. It is also true for all pairs below r as the points in this area were moved in a way that preserved the relative positions. Let us therefore assume that p' is below r and q' is on or above s. If p' is in $A(r)$ then the statement holds as p' was moved to the same vertical line which contained the corresponding point from $A(s)$ of M. Thus, there is a point $p \in A(r)$ with the desired relative position to q'. If $p' \notin A(r)$ and $\circ_d \in \{\nwarrow, \nearrow\}$ then the statement holds as p' was moved to a vertical stripe that has the same available upper witnesses in both directions as the stripe of q' had in M. It should be stressed here that the final ϵ-shift does not introduce new relationships between nodes in the same vertical stripe as for points that are not available witness points there are always available witness points in eastern *and* western direction. Finally, there can be no points $p' \notin A(r)$ and q' above s with $p' \uparrow q'$ by the final shift in the construction.

To show that all $\exists$-constraints are satisfied in M', we consider the witness function w of M. We say that a point p from M below r has a *remote witness with respect to a constraint* c if $w(p,c)$ is not below r, otherwise $w(p,c)$ is a *local witness*. Let p' be a point from M' below r, p its original point in M and $c \in C_\exists$. If $q = w(p,c)$ is a local witness then its corresponding point q' in M' is a witness for p'. If q is a remote witness we can assume without loss of generality that $q \in A(r)$. As the vertical stripe of p with respect to $A(r)$ is moved into the corresponding stripe with respect to $A(s)$, p' has a corresponding remote witness in $A(s)$.

The argument for witnesses to points above or on s is analogous. $\square$

Now we are ready to complete the proof of Proposition 6.

Proof. [of Proposition 6] Let $L = (\Sigma, C_\exists, C_\forall)$ be an instance of 2LPP($\mathcal{D}_-$). From Lemmas 7 and 8 we can infer that if L has a solution then it has one with at most $N =_{\text{def}} (6|\Sigma| + 1)!2^{6|\Sigma|}$ lines each of which contains at most $M =_{\text{def}} 2|\Sigma|$ points. Thus, altogether a possible minimal solution has at most $K =_{\text{def}} MN$ points. Hence, if there is a solution there is one with points from $G =_{\text{def}} \{1, \ldots, K\} \times \{1, \ldots, N\}$. However, such a set cannot be represented in polynomial at once.

To achieve the polynomial space bound, the algorithm follows the plane sweep paradigm. It sweeps over G from south to north and guesses, for each $r \in \{1, \ldots, N\}$, a profile A_r. To this end, in round $r + 1$, the algorithm guesses a profile A_{r+1} and checks that A_r and A_{r+1} are *consistent* with the semantics of profiles and that A_{r+1} is *valid* with respect to L.

More details can be found in the full version of the paper. □

3.4 A lower bound for $\mathrm{FO}^2(\leq, \precsim)$

The following result shows that the upper bound of Theorem 1 is sharp.

Theorem 9. *Finite satisfiability for $FO^2(\leq, \precsim)$ is* EXPSPACE*-hard.*

Proof. We reduce from a tiling problem. A *tile* is a square with colored edges. A *valid tiling* of an $m \times n$ grid with tiles from a tile-set T is a mapping $t : m \times n \to T$ such that adjacent edges have the same color, i.e., for example, the northern edge of $t(i, j)$ and the southern edge of $t(i, j + 1)$ are colored equally.

The following tiling problem is EXPSPACE-hard [Boa97]:

Problem: ExpCorridorTiling
Input: Tiles $T_1, \ldots, T_k$ over a set of colors $\{c_1, \ldots, c_m\}$, and a string 1^n.
Question: Is there a tiling of size $2^n \times l$ for $l \in \mathbb{N}$, such that the south of the bottom row as well as the north of the top row are colored c_1?

We reduce ExpCorridorTiling to $FO^2(\leq, \precsim)$. The rough idea is as follows. For a given tiling instance, we build an $FO^2(\leq, \precsim)$ formula φ such that positions of a valid tiling correspond to points in a model of φ. For encoding the column number of a tile we use unary relations $C_1, \ldots, C_n$, i.e. two tiles are in the same column iff they fulfill the same relations $C_1, \ldots, C_n$. For encoding rows of a tiling, we use equivalence classes of $\precsim$. In particular, two positions of a tiling are in the same row, if they are in the same equivalence class of $\precsim$. That is why we call equivalence classes of $\precsim$ rows, too.

The complete construction is given in the full paper. □

3.5 Undeciable Extensions

Theorem 10. *Finite satisfiability for $FO^2(\precsim_1, \precsim_2)$ is undecidable.*

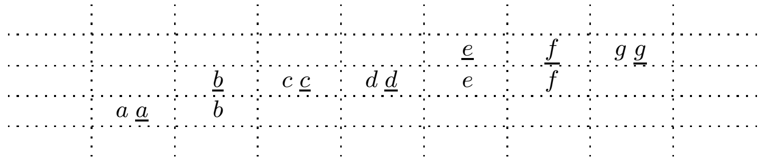

Fig. 3. How the valid sequences $u := ab|cdef|g$ and $v := a|bcd|efg$ are represented in a model for the $FO^2(\precsim_1, \precsim_2)$-formula φ. Rows represent equivalence classes of $\precsim_1$ and columns represent equivalence classes of $\precsim_2$. Letters from v are underlined.

Proof. We reduce from the Post Correspondence Problem:

Problem: PCP
Input: $(u_1, v_1), \ldots, (u_k, v_k) \in \Sigma^* \times \Sigma^*$
Question: Is there a non-empty sequence $i_1, \ldots, i_m$ such that $u_{i_1} \ldots u_{i_m} = v_{i_1} \ldots v_{i_m}$?

Let $(u_1, v_1), \ldots, (u_k, v_k)$ be an instance of the PCP over alphabet Σ. Firstly, we explain how an intended model, that is to say a valid sequence, shall be represented. Consider a valid sequence $(u_{i_1}, v_{i_1}), \ldots, (u_{i_m}, v_{i_m})$, i.e. a sequence such that $u := u_{i_1} \ldots u_{i_m} = v_{i_1} \ldots v_{i_m} =: v$. The intended model has one element, for every letter in the valid sequence, in particular, the size of the model is $2|u|$. The elements corresponding to the letters of a pair (u_{i_j}, v_{i_j}) are in one equivalence class E_{i_j} of $\precsim_1$. The equivalence classes of $\precsim_1$ are ordered as the pairs in the valid sequence, i.e. if $x \in E_{i_j}$, $y \in E_{i_k}$ and $j < k$ then $x \precsim_1 y$. All equivalence classes of $\precsim_2$ are of size 2 and contain exactly one element representing a letter of the u-sequence and its corresponding element for the v-sequence (and they should carry the same symbol). The equivalence classes of $\precsim_2$ are ordered as the sequences u and v. Figure 3 illustrates the construction for a simple example. The details of the proof can be found in the full version. □

Remark 11. In the above proof two elements p_1, p_2 can be equivalent with respect to both, $\precsim_1$ and $\precsim_2$. The proof can easily be extended for cases where no two points can be in the same equivalence class of both total preorders, for example in subsets of $\mathbb{Z} \times \mathbb{Z}$ with total preorders induced by the usual linear orders on x- and y-axis. Note, that in the above proof at most two elements can be in the same equivalence class of $\precsim_1$ and $\precsim_2$, and that one of them is labeled with U and the other one with V. An extended model can combine two such elements by allowing elements to carry both U and V labels at once. Then it has to be ensured that positions carrying both types of symbols are alone in their $\precsim_2$ equivalence class.

As the following result shows two linear orders and one total preorder yield undecidability as well..

Theorem 12. *Finite satisfiability for* $FO^2(\leq_1, \leq_2, \precsim)$ *is undecidable.*

4 Applications

In this section we outline how our result interacts with other well-known logics. We start with an informal discussion.

Data words extend conventional words by assigning data values to every position. Logics on data words then allow to use the usual relations for words as well as relations for the data values. We consider sets of data values with an order. It is known, that allowing the successor and order relations on positions of the word, as well as the order relation on data values, leads to an undecidable finite satisfiability problem. We prove that the finite satisfiability problem becomes decidable, when only the order relation on positions and order relation on data values are allowed.

Compass logic is a two-dimensional temporal logic, whose operators allow for moving north, south, east and west along a grid [Ven90]. Satisfibility for compass logic is known to be undecidable [MR97]. We extend compass logic in two directions. Upto now, only complete grids have been considered as underlying structure. We consider also partial grids as underlying structures, i.e. grids where not all crossings need to exist. Furthermore, we extend the model with the operators northeast, northwest, southeast and southwest. With these extensions, compass logic becomes decidable when only the operators northwest, northeast, southwest, southeast, west and east are used.

Interval temporal logic is a logic that can reason about intervals of time using operators as for example 'after', 'during', 'begins' etc. Expressions such as 'Immediately after we finnished writing the paper, we will go to the beach' can be captured. Therefore propositions, as for example "writing the paper" or "go to the beach", are assigned to time intervals. In conventional interval temporal logic, all intervals are part of a structure, i.e. reasoning always considers all intervals. We consider structures that are subsets of the set of all intervals and prove that satisfiability for reasoning with the operators 'ends', 'later' and 'during' as well as their duals, is decidable.

In both cases, compass logic and interval temporal logic, we show that the problems with more liberal semantics are at most as hard as the original problem.

4.1 Data Words

A *data word* is a finite sequence, in which every element (position) is labeled by a symbol from an alphabet Σ and a "data value" from some possibly infinite domain. For simplicity, we can assume that the domain consists of the set $\mathbb{N}$ of natural numbers. By $\mathrm{FO}^2(\Sigma, \leq, \precsim)$ we denote two-variable first-order logic with the binary relation symbols $\leq$, $\precsim$ and $a(x)$, for every $a \in \Sigma$. Data words can be seen as models for this logic. $a(x)$ is interpreted as 'there is an a at position x', $x < y$ is interpreted as 'x occurs before y' and $x \precsim y$ is interpreted as 'the data value of x is at most the data value of y'. We note that the same data value can appear more than once, whereas positions in the word are unique. We note further that data words can also be viewed as a set of points in the two-dimensional plane that are labeled with letters from Σ. Then the order on positions becomes a linear order on the x-axis and the order on the data values becomes a total preorder on the y-axis. Thus, the only difference between $FO^2(\leq, \precsim)$ and $\mathrm{FO}^2(\Sigma, \leq, \precsim)$ is, that in the latter the unary relations are always required to be a partition of the universe.

Given this connection, the following theorem follows readily from Theorem 1.

Theorem 13. *Finite satisfiability for* $\mathrm{FO}^2(\Sigma, \leq, \precsim)$ *is in* EXPSPACE.

However, the lower bound does not follow from Theorem 9 as the translation from $FO^2(\leq, \precsim)$ to $\mathrm{FO}^2(\Sigma, \leq, \precsim)$ might require an exponential size alphabet Σ. Thus, there remains a gap between this result and the NEXPTIME lower bound from [BMS+06].

Open question 1 *What is the exact complexity of finite satisfiability for* $\mathrm{FO}^2(\Sigma, \leq, \precsim)$*?*

4.2 Compass Logic

Due to lack of space we only state the main results on compass logic. The precise definitions and proofs can be found in the full paper.

Usually, compass logics is interpreted over grids $\langle \mathbb{D}_1 \times \mathbb{D}_2, \leq_1, \leq_2 \rangle$ that are the product of two linear orders $\langle \mathbb{D}_1, \leq_1 \rangle$ and $\langle \mathbb{D}_2, \leq_2 \rangle$. In this paper, we also consider *partial grids*, that is, subsets of grids. We show the following results.

Proposition 14. *Let $\mathcal{O}$ be a set of directional operators. If finite satisfiability of* $\mathrm{CL}(\mathcal{O})$ *is decidable on labeled grids, then it is decidable on labeled partial grids as well.*

Open question 2 *Is there a set $\mathcal{O}$ of operators, such that $CL(\mathcal{O})$ on partial grids is decidable, but $CL(\mathcal{O})$ on grids is undecidable?*

Theorem 15. *Finite satisfiability for* $\mathrm{CL}(\langle \rightarrow \rangle, \langle \leftarrow \rangle, \langle \nearrow \rangle, \langle \searrow \rangle, \langle \swarrow \rangle, \langle \nwarrow \rangle)$ *over partial grids is in* EXPSPACE.

4.3 Interval Temporal Logic

Similarly, as for compass logic, we consider interval logics over *partial interval structures*, that is subsets of the set of intervals over some linearly ordered set $\langle \mathbb{D}, \leq \rangle$. Detailed definitions and proofs are given in the full paper. We prove the following results.

Proposition 16. *Let $\mathcal{O}$ be a set of interval operators. If* $\mathrm{HS}(\mathcal{O})$ *is decidable over interval models, then* $\mathrm{HS}(\mathcal{O})$ *is decidable over partial interval models as well.*

Open question 3 *Is there a set $\mathcal{O}$ of operators, such that $HS(\mathcal{O})$ on partial interval models is decidable, but $HS(\mathcal{O})$ on interval models is undecidable?*

Theorem 17. *Finite satisfiability for* $\mathrm{HS}(\langle E \rangle, \langle \bar{E} \rangle, \langle L \rangle, \langle \bar{L} \rangle, \langle D \rangle, \langle \bar{D} \rangle)$ *over partial grid structures is in* EXPSPACE.

5 Conclusion

The context of our results was already discussed in the introduction and some open questions were stated in the body of the text. We mention some further open questions and possible lines of research.

- In the context of verification it would be interesting to generalize our results from data words to data ω-words
- There is still a gap between the "eight orders" undecidability result of [Ott01] and the decidability for FO^2 with two linear orders in this paper.
- We believe that partial models for compass and interval logic deserve some further investigations.

References

[BMS+06] Mikolaj Bojanczyk, Anca Muscholl, Thomas Schwentick, Luc Segoufin, and Claire David, *Two-variable logic on words with data*, LICS, 2006, pp. 7–16.

[Boa97] Peter Van Emde Boas, *The convenience of tilings*, In Complexity, Logic, and Recursion Theory, Marcel Dekker Inc, 1997, pp. 331–363.

[CH01] Anthony G. Cohn and Shyamanta M. Hazarika, *Qualitative spatial representation and reasoning: An overview*, Fundam. Inform. **46** (2001), no. 1-2, 1–29.

[DL09] Stéphane Demri and Ranko Lazic, *Ltl with the freeze quantifier and register automata*, ACM Trans. Comput. Log. **10** (2009), no. 3, 16:1–16:30.

[FS09] Diego Figueira and Luc Segoufin, *Future-looking logics on data words and trees*, MFCS, 2009, pp. 331–343.

[GKV97] Erich Grädel, Phokion G. Kolaitis, and Moshe Y. Vardi, *On the decision problem for two-variable first-order logic*, Bulletin of Symbolic Logic **3** (1997), no. 1, 53–69.

[GO99] Erich Grädel and Martin Otto, *On logics with two variables*, Theor. Comput. Sci. **224** (1999), no. 1-2, 73–113.

[KO05] Emanuel Kieronski and Martin Otto, *Small substructures and decidability issues for first-order logic with two variables*, LICS, 2005, pp. 448–457.

[KT09] Emanuel Kieronski and Lidia Tendera, *On finite satisfiability of two-variable first-order logic with equivalence relations*, LICS, 2009, pp. 123–132.

[Man10] Amaldev Manuel, Personal communication, 2010.

[Mor75] M. Mortimer, *On languages with two variables*, Zeitschr. f. math. Logik u. Grundlagen d. Math. **21** (1975), 135–140.

[MR97] Maarten Marx and Mark Reynolds, *Undecidability of compass logic*, Journal of Logic and Computation **9** (1997), 897–941.

[Ott01] Martin Otto, *Two variable first-order logic over ordered domains*, J. Symb. Log. **66** (2001), no. 2, 685–702.

[Seg06] Luc Segoufin, *Automata and logics for words and trees over an infinite alphabet*, CSL, 2006, pp. 41–57.

[Ven90] Yde Venema, *Expressiveness and completeness of an interval tense logic*, Notre Dame Journal of Formal Logic **31** (1990), no. 4, 529–547.